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Cosmological singularities and Bel–Robinson energy

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Abstract

We consider the problem of describing the asymptotic behaviour of FRW universes near their spacetime singularities in general relativity. We find that the Bel–Robinson energy of these universes in conjunction with the Hubble expansion rate and the scale factor proves to be an appropriate measure leading to a complete classification of the possible singularities. We show how our scheme covers all known cases of cosmological asymptotics possible in these universes and also predicts new and distinct types of singularities. We further prove that various asymptotic forms met in flat cosmologies continue to hold true in their curved counterparts. These include phantom universes with their recently discovered big rips, sudden singularities as well as others belonging to graduated inflationary models.

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1. Introduction

The problem of cosmological singularities is perhaps the single most important unsolved issue in modern theoretical general relativity and cosmology. It is usually formulated as being a two-faceted problem, the first aspect being the issue of finding conditions under which singularities are expected to develop in a finite time (existence) during the evolution of a model spacetime, the second being that of describing their character or nature. In fact, the other, closely related, issue in general relativity, that of proving geodesic completeness, spacetimes existing for an infinite proper time interval, is usually thought of as the negation of the singularity problem; cf. [1]. The geodesic completeness problem may also be formulated as a two-sided issue, one aspect being, like the singularity problem, that of finding conditions that guarantee the existence of complete spacetimes, the other being that of unraveling the nature of geodesic completeness, that is the nature of behaviour of the controlling functions, for instance, their decay to infinity.

The issue of proving existence of singular and/or complete spacetimes is basically a geometric one. Traditionally one is interested in formulating criteria, of a geometric nature, to test under what circumstances singularities, in the

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form of geodesic incompleteness, will be formed during the evolution in general relativity and in other metric theories of gravity; cf. [2]. These, apart from the causality ones like global hyperbolicity, are usually translated into sufficient conditions to be satisfied by the matter fields present and are very plausible. Such conditions guarantee that in a finite time in the future or past of the spacetime, some quantity, usually describing the convergence of a bundle of geodesics, blows up, thus designating the past or future breakdown of the bundle. On the other hand, one can formulate equally plausible and generic geometric criteria for the long time existence of geodesically complete, generic spacetimes in general relativity and other theories of gravity; cf. [3]. Such sufficient conditions assume a globally hyperbolic spacetime in the so-called generic sliced form (cf. [4]), require the space gradient of the lapse function as well as the extrinsic curvature not to grow without bound, and are also very plausible. They provide us with assumptions under which generic spacetimes will exist forever. All known global-in-time spacetimes satisfy these sufficient conditions and are therefore future geodesically complete.

Of course as the singularity theorems do not detail the nature of the behaviour of the controlling functions on approach to the singularity, similarly the completeness theorems do not provide conditions for, say, the decay of such functions to infinity. A lesson to be drawn from this state of affairs is that in general relativity (and in other metric gravity theories), singular and complete spacetimes are equally generic in a sense. The real question is what do we mean when we say that a relativistic model develops a spacetime singularity during its evolution. In other words, what are the possible spacetime singularities which are allowed in gravity theories? These may be singularities in the form of geodesic incompleteness developing in the course of the evolution but can be also others which are more subtle and which are also dynamical and present themselves, for instance, in some higher derivatives of the metric functions spoiling the smoothness of global solutions to the field equations.

Obviously a recognition and complete analysis of such a program cannot be accomplished in the short run and requires complete examination of a number of different factors controlling the resulting behaviour. For instance, one needs to have control of each possible behaviour of the different families of relativistic geometry coupled to matter fields in general relativity and other theories of gravity to charter the possible singularity formation. In this sense, unraveling the nature and kinds of possible singularities in the simplest kinds of geometry becomes as important as examining this problem in the most general solutions to the Einstein equations. In fact, an examination of the literature reveals that the more general the spacetime geometry considered (and thus the more complex the system of equations to be examined) the simpler is the type of singularities allowed to be examined. By starting with a simple cosmological spacetime we allow all possible types of singularities to come to the surface and be analyzed.

In [5] we derived necessary conditions for the existence of finite time singularities in globally and regularly hyperbolic isotropic universes, and provided the first evidence for their nature based entirely on the behaviour of the Hubble parameter (extrinsic curvature). This analysis exploited directly the evidence provided by the completeness theorems proved in [3]. The main result proved in [5] may be summarized as follows:

Theorem 1.1. *Necessary conditions for the existence of finite time singularities in globally hyperbolic, regularly hyperbolic* FRW *universes are:*

S_1 *For each finite t, H is non-integrable on $[t_1, t]$, or*
S_2 *H blows up in a finite future time, or*
S_3 *H is defined and integrable for only a finite proper time interval.*

Condition S_1 may describe different types of singularities. For instance, it describes a big bang type of singularity, when H blows up at t_1, since then it is not integrable on any interval of the form $[t_1, t]$, $t > t_1$ (regular hyperbolicity is violated in this case, but the scale factor is bounded from above). However, under S_1 we can have other types of singularities: Since $H(\tau)$ is integrable on an interval $[t_1, t]$, if $H(\tau)$ is defined on $[t_1, t]$, continuous on (t_1, t) and the limits $\lim_{\tau\to t_1^+} H(\tau)$ and $\lim_{\tau\to t^-} H(\tau)$ exist, the violation of *any* of these conditions leads to a singularity that is not of the big bang type discussed previously.

Condition S_2 describes a future blow-up singularity and condition S_3 may lead to a sudden singularity (where H remains finite), but for this to be a genuine type of singularity, in the sense of geodesic incompleteness, one needs to demonstrate that the metric is non-extendible to a larger interval.

Note that these three conditions are not overlapping, for example S_1 is not implied by S_2 for if H blows up at some finite time t_s after t_1, then it may still be integrable on $[t_1, t]$, $t_1 < t < t_s$. As discussed in [5], Theorem 1.1 describes possible time singularities that are met in FRW universes having a Hubble parameter that behaves like S_1 or S_2 or S_3.

Although such a classification is a first step to clearly distinguishing between the various types of singularities that can occur in such universes, it does not bring out some of the essential features of the dynamics that differ from singularity to singularity. For instance, condition S_2 includes both a collapse singularity, where the scale factor $a \to 0$ as $t \to t_s$, *and* a blow-up singularity where $a \to \infty$ as $t \to t_s$. Such a degeneration is unwanted in any classification of the singularities that can occur in the model universes in question.

It is therefore necessary to extend and refine this classification by considering also the behaviour of the scale factor. That it is only necessary to include this behaviour is also seen most clearly by noticing that only in this way may one consistently distinguish between initial and final singularities, or between past and future ones, as has been repeatedly emphasized by Penrose over the years in his Weyl curvature hypothesis; cf. [6]. Another aspect of the problem that must be taken into account is the relative behaviour of the various matter components as we approach the time singularities. We believe that a clear picture of the behaviour of the matter fields on approach to a singularity will contribute to an understanding of its nature.

In this paper we present a new scheme that describes in a consistent way the possible spacetime singularities which can form during the evolution of the FRW cosmological models in general relativity. Section 2 introduces the classification and shows that by extending our previous results and taking into account the effects induced by the Bel–Robinson energy, we can arrive at the possible forms of singular behaviour that the universes in question can exhibit. Section 3 proves that various singularity types predicted by our scheme have a general significance in the isotropic category and also describes how some singularity types known to occur only in flat isotropic universes do in fact appear in many curved spacetimes. In the last section we discuss various aspects of this work including the issue of how to describe known types of singularities with our technique.

2. Classification

In this section we present a classification of possible singularities which occur in isotropic model universes. Our classification is based on the introduction of an invariant geometric quantity, the Bel–Robinson energy (cf. [7] and references therein), which takes into account precisely those features of the problem, related to the matter contribution, in which models still differ near the time singularity while having similar behaviours of a and H. In this way, we arrive at a complete classification of the possible cosmological singularities in the isotropic case.

The Bel–Robinson energy is a kind of energy of the gravitational field *projected* in a sense to a slice in spacetime. It is used in [8,9] to prove global existence results in the case of asymptotically flat and cosmological spacetimes respectively, and is defined as follows. Consider a sliced spacetime with metric

$$^{(n+1)}g \equiv -N^2(\theta^0)^2 + g_{ij}\theta^i\theta^j, \quad \theta^0 = dt,\ \theta^i \equiv dx^i + \beta^i dt, \tag{2.1}$$

where $N = N(t, x^i)$ is the lapse function and $\beta^i(t, x^j)$ is the shift function, and the 2-covariant spatial electric and magnetic tensors are

$$E_{ij} = R^0_{i0j},$$
$$D_{ij} = \frac{1}{4}\eta_{ihk}\eta_{jlm}R^{hklm},$$
$$H_{ij} = \frac{1}{2}N^{-1}\eta_{ihk}R^{hk}_{0j},$$
$$B_{ji} = \frac{1}{2}N^{-1}\eta_{ihk}R^{hk}_{0j},$$

where η_{ijk} is the volume element of the space metric $\bar{g}$. These four time-dependent space tensors comprise what is called a *Bianchi field*, (E, H, D, B), a very important frame field used to prove global in time results; cf. [7].

The *Bel–Robinson energy at time t* is given by

$$\mathcal{B}(t) = \frac{1}{2}\int_{\mathcal{M}_t}(|E|^2 + |D|^2 + |B|^2 + |H|^2)\mathrm{d}\mu_{\bar{g}_t}, \tag{2.2}$$

where by $|X|^2 = g^{ij}g^{kl}X_{ik}X_{jl}$ we denote the spatial norm of the 2-covariant tensor X. In the following, we exclusively use an FRW universe filled with various forms of matter with metric given by

$$ds^2 = -dt^2 + a^2(t)d\sigma^2, \tag{2.3}$$

where $d\sigma^2$ denotes the usual time-independent metric on the 3-slices of constant curvature k. For this spacetime, we find that the norms of the magnetic parts, $|H|$, $|B|$, are identically zero while $|E|$ and $|D|$, the norms of the electric parts, reduce to the forms

$$|E|^2 = 3(\ddot{a}/a)^2 \quad \text{and} \quad |D|^2 = 3((\dot{a}/a)^2 + k/a^2)^2. \tag{2.4}$$

Therefore the Bel–Robinson energy becomes

$$\mathcal{B}(t) = \frac{C}{2}(|E|^2 + |D|^2), \tag{2.5}$$

where C is the constant volume of (or *in* in the case of a non-compact space) the three-dimensional slice at time t.

It is not difficult to show that for a closed, RW universe such that $|D|$ is bounded above, H must be bounded above and the scale factor bounded below. Therefore H must be integrable and the spatial metric bounded below, that is such a universe is regularly hyperbolic. This in turn means that all hypotheses of the completeness theorem proved in [3] are satisfied and therefore such a universe is g-complete. It is also straightforward to see that the null energy condition is equivalent to the inequality $|E| \leq |D|$, and hence completeness is then accompanied with $|E|$ being bounded above. (For a flat spatial metric we would need to impose the regular hyperbolicity hypothesis in order to conclude completeness since the latter is independent from the boundedness of $|D|$.)

We can now proceed to list the possible types of singularities that are formed in an FRW geometry during its cosmic evolution and enumerate the possible types that result from the different combinations of the three main functions in the problem, namely, the scale factor a, the Hubble expansion rate H and the Bel–Robinson energy $\mathcal{B}$. These types will by necessity entail a possible blow-up in the functions $|E|$, $|D|$. If we suppose that the model has a finite time singularity at $t = t_s$, then the possible behaviours of the functions in the triplet $(H, a, (|E|, |D|))$ in accordance with Theorem 1.1 are as follows:

S_1 H non-integrable on $[t_1, t]$ for every $t > t_1$,
S_2 $H \to \infty$ at $t_s > t_1$,
S_3 H otherwise pathological,
N_1 $a \to 0$,
N_2 $a \to a_s \neq 0$,
N_3 $a \to \infty$,
B_1 $|E| \to \infty, |D| \to \infty$,
B_2 $|E| < \infty, |D| \to \infty$,
B_3 $|E| \to \infty, |D| < \infty$,
B_4 $|E| < \infty, |D| < \infty$.

The nature of a prescribed singularity is thus described completely by specifying the components in a triplet of the form

$$(S_i, N_j, B_l),$$

with the indices i, j, l taking their respective values as above.

Note that there are a few types that cannot occur. For instance, we cannot have an (S_2, N_2, B_3) singularity because that would imply having $a < \infty$ (N_2) and $H \to \infty$ (S_2), while $3((\dot{a}/a)^2 + k/a^2)^2 < \infty$ (B_3), at t_s which is impossible since $|D|^2 \to \infty$ at t_s (k arbitrary).

A complete list of impossible singularities is model dependent and is generally given by triplets (S_i, N_j, B_l), where the indices, in the case of a $k = 0, +1$ universe, take the values $i = 1, 2$, $j = 1, 2, 3$, $l = 3, 4$, whereas in the case of a $k = -1$ universe the indices take the values $i = 1, 2$, $j = 2, 3$, $l = 3, 4$ (here by S_1 we denote for simplicity only the big bang case in the S_1 category). We thus see that some singularities which are impossible for a flat or a closed universe become possible for an open universe. Consider for example the triplet (S_2, N_1, B_3) which means having $H \to \infty$, $a \to 0$ and

$$|D|^2 = 3((\dot{a}/a)^2 + k/a^2)^2 < \infty$$

at t_s. This behaviour is valid only for some cases of an open universe.

All other types of finite time singularities can in principle be formed during the evolution of FRW, matter-filled models, in general relativity or other metric theories of gravity.

It is interesting to note that all the standard dust or radiation-filled big bang singularities fall under the *strongest* singularity type, namely, the type (S_1, N_1, B_1). For example, in a flat universe filled with dust, at $t = 0$ we have

$$a(t) \propto t^{2/3} \to 0, \quad (N_1), \tag{2.6}$$

$$H \propto t^{-1} \to \infty, \quad (S_1), \tag{2.7}$$

$$|E|^2 = 3/4H^4 \to \infty, \quad |D|^2 = 3H^4 \to \infty, \quad (B_1). \tag{2.8}$$

Note that our scheme is organized in such a way that the character of the singularities (i.e., the behaviour of the defining functions) becomes milder as the indices of S, N and B increase. Milder singularities in isotropic universes are thus expected to occur as one proceeds down the singularity list.

It is the purpose of this classification to apply both to vacuum as well as matter dominated models. In fact the Bel–Robinson energy takes care in a very neat way of the matter case. For instance, in fluid-filled models, the various behaviours of the Bel–Robinson energy density can be related to four conditions imposed on the density and pressure of the cosmological fluid:

$B_1 \Leftrightarrow \mu \to \infty$ and $|\mu + 3p| \to \infty$,
$B_2 \Leftrightarrow \mu \to \infty$ and $|\mu + 3p| < \infty$,
$B_3 \Leftrightarrow \mu < \infty$ and $|\mu + 3p| \to \infty \Leftrightarrow \mu < \infty$ and $|p| \to \infty$,
$B_4 \Leftrightarrow \mu < \infty$ and $|\mu + 3p| < \infty \Leftrightarrow \mu < \infty$ and $|p| < \infty$.

Of course we can translate these conditions to asymptotic behaviours in terms of a, H, depending on the value of k, for example,

1. If $k = 0$, $\mu < \infty \Rightarrow H^2 < \infty$, a arbitrary,
2. If $k = 1$, $\mu < \infty \Rightarrow H^2 < \infty$ and $a \neq 0$,
3. If $k = -1$, $\mu < \infty \Rightarrow H^2 - 1/a^2 < \infty$.

As an example, we consider the sudden pressure singularity introduced by Barrow in [10]. This has a finite a (condition N_2), finite H (condition S_3), finite μ but a divergent p (condition B_3) at t_s.

As another example, consider the flat FRW model containing dust and a scalar field studied in [11]. The scale factor collapses at both an initial (big bang) and a final time (big crunch). The Hubble parameter and $\ddot{a}/a$ both blow up at the times of the big bang and big crunch (cf. [5]) leading to an (S_1, N_1, B_1) big bang singularity and an (S_2, N_1, B_1) big crunch singularity, respectively.

3. Generic results and examples

It was recently shown by Ellis in [12] that a RW space with scale factor $a(t)$ admits a past closed trapped surface if the following condition is satisfied:

$$\dot{a}(t) > \left| \frac{f'(r)}{f(r)} \right|, \tag{3.1}$$

with $f(r) = \sin r, r, \sinh r$ for $k = 1, 0, -1$ respectively. Recall that a closed trapped surface is a 2-surface with spherical topology such that both families of incoming and outgoing null geodesics orthogonal to the surface converge. Since our function $|D|$ can be written in the form

$$|D| = \frac{\sqrt{3}}{a^2(t)} \left| \left(\dot{a}(t) - \frac{f'(r)}{f(r)} \right) \left(\dot{a}(t) + \frac{f'(r)}{f(r)} \right) + \frac{1}{f^2(r)} \right|, \tag{3.2}$$

we see that the condition for the existence of a closed trapped surface becomes equivalent to the following inequality:

$$|D| > \frac{\sqrt{3}}{a^2(t) f^2(r)}. \tag{3.3}$$

We thus conclude that collapse singularities (as predicted by the existence of a trapped surface) are characterized by a divergent Bel–Robinson energy.

In this section we provide necessary and sufficient conditions for the occurrence of some of the triplets detailing the nature of the singularities introduced above. These conditions are motivated from studies of cosmological models described by exact solutions in the recent literature. By exact solutions we mean those in which all arbitrary constants have been given fixed values. We expect the proofs of these results to be all quite straightforward, for we have now *already* identified the type of singularity that we are looking for in accordance with our classification. Proving such results *without* this knowledge would have been a problem of quite a different order.

The usefulness of the results proved below lies in that they answer the question of whether the behaviours met in known models described by exact solutions (which as a rule have a *flat* spatial metric ($k = 0$)) continue to be valid in universes having non-zero values of k as well as described by solutions which are more general than exact in the sense that some or all of the arbitrary constants present remain arbitrary. The reason behind this behaviour, whenever it is met, lies in the fact that the curvature term in the Friedmann equation turns out to be usually subdominant compared to the density term or in any case cannot alter the H behaviour. For the purpose of organization we present separately results about future and past singularities.

3.1. Future singularities

The first result of this subsection characterizes the future singularity in phantom cosmologies *irrespective* of the value of the curvature k and says that the singularities in such models can be milder than the standard big crunches and have necessarily diverging pressure and the characteristic “phantom” equation of state.

Theorem 3.1. *Necessary and sufficient conditions for an* (S_2, N_3, B_1) *singularity occurring at the finite future time* t_s *in an isotropic universe filled with a fluid with equation of state* $p = w\mu$ *are that* $w < -1$ *and* $|p| \to \infty$ *at* t_s.

Proof. Substituting the equation of state $p = w\mu$ in the continuity equation $\dot{\mu} + 3H(\mu + p) = 0$, we have

$$\mu \propto a^{-3(w+1)}, \tag{3.4}$$

and so if $w < -1$ and p blows up at t_s, a also blows up at t_s. Since

$$H^2 = \frac{\mu}{3} - \frac{k}{a^2}, \qquad |D|^2 = \frac{\mu^2}{3}, \qquad |E|^2 = \frac{1}{12}\mu^2(1+3w)^2, \tag{3.5}$$

we conclude that at t_s, H, a, $|D|$ and $|E|$ are divergent.

Conversely, assuming an (S_2, N_3, B_1) singularity at t_s in an FRW universe with the equation of state $p = w\mu$, we have from the (B_1) hypothesis that $\mu \to \infty$ at t_s and so p also blows up at t_s. Since a is divergent as well, we see from (3.4) that $w < -1$. ■

As an example, consider an exact solution which describes a flat, isotropic phantom dark-energy-filled universe, studied in [13] given by

$$\alpha = \left[{\alpha_0}^{3(1+w)/2} + \frac{3(1+w)\sqrt{A}}{2}(t - t_0) \right]^{\frac{2}{3(1+w)}}, \tag{3.6}$$

where A is a constant. From (3.4) we see that the scale factor, and consequently H, blows up at the finite time

$$t_s = t_0 + \frac{2}{3\sqrt{A}(|w|-1)\alpha_0^{3(|w|-1)/2}}.$$

Then it follows that $|E|^2 = \frac{3}{4}H^4(1+3w)^2$ and $|D|^2 = 3H^4$ also blow up at t_s. Therefore in this model the finite time singularity is of type (S_2, N_3, B_1).

Next we focus on a generalization of a model, called graduated inflation and originally proposed by Barrow — see the next subsection, given in [14]. Consider a flat FRW filled with a fluid with equation of state $p + \mu = -B\mu^{\beta}$,

$\beta > 1$. As $\mu \to \infty$,

$$t \to t_0 + \frac{2}{\sqrt{3}\kappa B}\frac{\mu^{-\beta+1/2}}{1-2\beta} \to t_0,$$

where $\kappa^2 = 8\pi G$ and t_0 is an integration constant. It follows then that $|p| \to \infty$ at t_0. The scale factor is described by (3.7) below and it is therefore finite. However, H, $|D|$ and $|E|$ all diverge at t_0 meaning that this is an (S_2, N_2, B_1) singularity in our scheme. As we shall show in the next subsection, the introduction of a more general equation of state has resulted in "taming" the singularity of the graduated inflationary type. This behaviour is a special case of the following general result which holds also in curved models.

Theorem 3.2. *A necessary and sufficient condition for an (S_2, N_2, B_1) singularity at t_s in an isotropic universe filled with a fluid with equation of state $p + \mu = -B\mu^\beta$, $\beta > 1$, is that $\mu \to \infty$ at t_s.*

Proof. From the continuity equation we have

$$a = a_0 \exp\left(\frac{\mu^{1-\beta}}{3B(1-\beta)}\right), \tag{3.7}$$

and so $a \to a_0$ as $t \to t_s$. Since

$$H^2 = \frac{\mu}{3} - \frac{k}{a^2}, \qquad |E|^2 = \frac{1}{12}(2\mu + 3B\mu^\beta)^2, \qquad |D|^2 = \frac{\mu^2}{3}, \tag{3.8}$$

we see that as $t \to t_s$, H, $|E|$ and $|D|$ diverge, leading precisely to an (S_2, N_2, B_1) singularity. The converse is obvious. ■

Similarly one can prove:

Theorem 3.3. *A necessary and sufficient condition for an (S_3, N_2, B_3) singularity at t_s in an isotropic universe filled with a fluid with equation of state $p + \mu = -C(\mu_0 - \mu)^{-\gamma}$, $\gamma > 0$, is that $\mu \to \mu_0$ at t_s.*

Proof. Again using the continuity equation we find

$$a \propto \exp\left\{-\frac{(\mu_0 - \mu)^{\gamma+1}}{3C(\gamma+1)}\right\}, \tag{3.9}$$

which is finite as $t \to t_s$. Also since

$$H^2 = \frac{\mu}{3} - \frac{k}{a^2}, \qquad |E|^2 = \frac{1}{12}(2\mu + 3C(\mu_0 - \mu)^{-\gamma})^2, \qquad |D|^2 = \frac{\mu^2}{3}, \tag{3.10}$$

we see that as $t \to t_s$, H and $|D|$ remain finite whereas $|E|$ diverges, leading to an (S_3, N_2, B_3) singularity. The converse is immediate. ■

The above equation of state is studied in [14] for the case of a flat universe. It follows then that as $\mu \to \mu_0$, $t = t_0 - \frac{(\mu_0-\mu)^{\gamma+1}}{\kappa C\sqrt{3\mu_0}(\gamma+1)} \to t_0$ (an integration constant) and $|p| \to \infty$. The scale factor is described by (3.9) and it is therefore finite at t_0 whereas $H < \infty$, $|D| < \infty$ and $|E| \to \infty$ leading to an (S_3, N_2, B_3) singularity. We therefore see that the nature of the singularity depends very sensitively on even very mild changes in the equation of state.

A way to become more intimately acquainted with the nature of the various singularities and to clearly distinguish their various differences is to study the relative asymptotic behaviours of the three functions that define the type of singularity. Using a standard notation that expresses the behaviour of two functions around the singularity at t_*, we can introduce a kind of relative "strength" in the singularity classification. Let f, g be two functions. We say that

1. $f(t)$ is *much smaller* than $g(t)$, $f(t) \ll g(t)$, if and only if $\lim_{t\to t_*} f(t)/g(t) = 0$,
2. $f(t)$ is *similar* to $g(t)$, $f(t) \sim g(t)$, if and only if $0 < \lim_{t\to t_*} f(t)/g(t) < \infty$,
3. $f(t)$ is *asymptotic* to $g(t)$, $f(t) \leftrightarrow g(t)$, if and only if $\lim_{t\to t_*} f(t)/g(t) = 1$,

Using these notions we find that standard radiation-filled isotropic universes (with $k = 0, \pm 1$) have the asymptotic behaviours described by $a \ll H \ll (|E| \leftrightarrow |D|)$, whereas the rest of the standard big bang singularities have $a \ll H \ll (|E| \sim |D|)$.

The phantom model of Theorem 3.1 has three possible behaviours depending on the ranges of the w parameter, namely, if $-5/3 < w < -4/3$ we have $H \ll a \ll (|E| \sim |D|)$, if $-4/3 < w < -1$ then $H \ll (|E| \sim |D|) \ll a$ whereas if $w < -5/3$ we have $a \ll H \ll (|E| \sim |D|)$.

The fluid case of Theorem 3.2 has the behaviour $a \ll H \ll |D| \ll |E|$.

The sudden singularity of Theorem 3.3 is characterized by the behaviour $(H \sim |D| \sim a) \ll |E|$. If we identify a sudden singularity as the kind of singularity occurring with a divergent p while a and H remain finite, then this asymptotic behaviour is the only possibility.

In the majority of the types of singularities that we have met in our study the two quantities $|E|, |D|$ are the most wildly diverging functions and therefore become the most dominant ones asymptotically.

3.2. Past singularities

We now present two results on the nature of past time singularities. These include the standard big bang ones but the latter are obviously *not* the only possibility. The first result given below predicts that the time singularity met in an exact solution in the family of the so-called graduated inflationary models first constructed in [15] extends also to open universes. The model consists of a flat isotropic spacetime with a fluid with equation of state $p + \mu = \gamma \mu^{3/4}$, $\gamma > 0$, and admits the exact solution

$$a = \exp\left(-\frac{16}{3^{3/2}\gamma^2 t}\right), \quad \text{with } \mu = \frac{256}{9\gamma^4 t^4}. \tag{3.11}$$

At $t = 0$, $a \to 0$ (N_1), $H \to \infty$ (S_1) and $|E|^2 = \frac{1}{12}(-2\mu + 3\gamma\mu^{3/4})^2 \to \infty$, $|D|^2 \to \infty (B_1)$. This behaviour can be made to occur more generally due to the following result.

Theorem 3.4. *A necessary and sufficient condition for an (S_1, N_1, B_1) singularity at t_1 in an open or flat isotropic universe filled with a fluid with equation of state $p + \mu = \gamma\mu^{\lambda}$, $\gamma > 0$ and $\lambda < 1$, is that $\mu \to \infty$ at t_1.*

Proof. The continuity equation gives

$$a = a_0 \exp\left(\frac{\mu^{-\lambda+1}}{3\gamma(\lambda - 1)}\right), \tag{3.12}$$

so that $a \to 0$ as $t \to t_1$. Since

$$H^2 = \frac{\mu}{3} - \frac{k}{a^2} = \frac{\mu}{3} - k{a_0}^{-2} \exp\left(\frac{-2\mu^{-\lambda+1}}{3\gamma(\lambda - 1)}\right) > 0,$$
$$|E|^2 = \frac{1}{12}(-2\mu + 3\gamma\mu^{\lambda})^2,$$
$$|D|^2 = \frac{\mu^2}{3},$$

we see that as $t \to t_1$ H, $|E|$ and $|D|$ diverge provided that $k = 0$ or $k = -1$. The converse is straightforward. ■

The asymptotic strength of this singularity is $a \ll (|E| \leftrightarrow |D|) \ll H$.

The second result that we prove now says that the strongest big bang type singularities in universes with a massless scalar field are produced due to the special form of the kinetic term. An exact solution for the case of a flat spatial metric is derived in [16] and is given by

$$H = \frac{1}{3t}, \qquad \phi = \pm\sqrt{\frac{2}{3}} \ln \frac{t}{c}.$$

Since $a \propto t^{1/3}$ we have that at $t = 0$, $a \to 0$ (N_1), $H = 1/(3t) \to \infty$ (S_1), $|E|^2 = \dot{\phi}^4/3 \to \infty$ and $|D|^2 = 3H^4 \to \infty$ (B_1). As follows from [16], this exact solution represents the asymptotic behaviour of a scalar field model if

$$\lim_{\phi\to\pm\infty} \mathrm{e}^{-\sqrt{6}|\phi|}V(\phi) = 0.$$

Theorem 3.5. *A necessary and sufficient condition for an (S_1, N_1, B_1) singularity at t_1 in an isotropic universe with a massless scalar field is that $\dot{\phi} \to \infty$ at t_1.*

Proof. From the continuity equation, $\ddot{\phi} + 3H\dot{\phi} = 0$, we have $\dot{\phi} \propto a^{-3}$, and if $\dot{\phi} \to \infty$ then $a \to 0$. Since

$$H^2 = \frac{\mu}{3} - \frac{k}{a^2} \to \infty \tag{3.13}$$

H becomes unbounded at t_1. In addition, since

$$|D|^2 = \frac{\mu^2}{3} = \frac{\dot{\phi}^4}{12} \to \infty, \tag{3.14}$$

and

$$|E|^2 = \frac{1}{12}(\mu + 3p)^2 = \frac{\dot{\phi}^4}{3} \to \infty, \tag{3.15}$$

at t_1, both $|D|$ and $|E|$ diverge there.

Conversely, assuming an (S_1, N_1, B_1) singularity at t_1, we have (from B_1) that $\mu \to \infty$ and so $\dot{\phi}^2 \to \infty$ at t_1. ■

The asymptotic strength of the singularity for the model described by both the exact solution and the above theorem is $a \ll H \ll (|E| \sim |D|)$.

A very mild type of singularity was discovered recently in [14]. We shall see how our classification is able to pick this up. The model is that of a flat FRW type filled with a fluid with equation of state

$$p + \mu = -\frac{AB\mu^{2\beta-1}}{A\mu^{\beta-1} + B}, \quad 0 < \beta < 1/2,$$

and for $\beta = 1/5$ it admits the exact solution $a = a_0 \mathrm{e}^{\tau^{8/3}}$. Then $H = (8/3)\tau^{5/3}$, $\dot{H} = (40/9)\tau^{2/3}$ and $\ddot{H} = (80/27)\tau^{-1/3}$. Thus as $\tau \to 0$, a, $\dot{a}$, $\ddot{a}$, H, $\dot{H}$ all remain finite whereas $\ddot{H}$ becomes divergent.

We can easily see that in this universe the Bel–Robinson energy at the initial time, $\mathcal{B}(0)$, is finite whereas *its time derivative* is

$$\dot{\mathcal{B}}(\tau) = 3\left[2\frac{\ddot{a}}{a}(\ddot{H} + 2H\dot{H}) + 4\left(\frac{k}{a^2} + H^2\right)\left(-\frac{kH}{a^2} + H\dot{H}\right)\right] \tag{3.16}$$

and thus $\dot{\mathcal{B}}(\tau) \to \infty$ at $\tau \to 0$. This is like a B_4 singularity in our scheme. Since the derivative of the Bel–Robinson energy diverges, we may interpret this singularity geometrically as one *in the velocities of the Bianchi (frame) field.* At t_s the Bianchi field encounters a cusp and its velocity diverges there. We believe this to be the mildest type of singularity known to date.

4. Discussion

In this paper we have extended and refined the classification of the cosmological singularities possible in an FRW universe in general relativity. A classification based entirely on the asymptotic behaviour of the scale factor or the Hubble expansion rate cannot lead to complete results and we have found it necessary to extend this scheme to include the behaviour of the Bel–Robinson energy. In this case we have found that the resulting behaviours of the three functions, H, a and $\mathcal{B}$ taken together exhaust the types of singularities that are possible to form during the evolution of an isotropic universe.

The resulting classification is described by triples of the form (S_i, N_j, B_l) where the S category monitors the asymptotic behaviour of the expansion rate, closely related to the extrinsic curvature of the spatial slices, N that of the scale factor, describing in a sense what the whole of space eventually does, while B describes how the matter fields contribute to the evolution of the geometry on approach to the singularity. We know (cf. [3,7]) that all these quantities need to be uniformly bounded to produce geodesically complete universes. Otherwise, the whole situation can be very complicated and we have exploited what can happen in such a case when we consider a relatively simple geometry as we have done in this paper.

Our scheme not only covers all the recently discovered types of singularities but also predicts many possible new ones. For example, in the case of a flat isotropic universe the classification of [14] provides us with four main types of singularity. These four types can be identified with four particular (S_i, N_j, B_k) triplets of our scheme. The 'big rip' type characterized by $a \to \infty$, $\mu \to \infty$ and $|p| \to \infty$ at t_s is an (S_2, N_3, B_1) singularity; the 'sudden' singularity described by $a \to a_s < \infty$, $\mu \to \mu_s < \infty$ and $|p| \to \infty$ at t_s is an (S_3, N_2, B_3); further, the so-called type III singularity, namely $a \to a_s < \infty$, $\mu \to \infty$ and $|p| \to \infty$ at t_s is clearly an (S_2, N_2, B_1) singularity, while type IV singularities with $a \to a_s < \infty$, $\mu \to 0$ and $|p| \to 0$ at t_s belong to an (S_3, N_2, B_4).

Having all possible singularity types in the form expounded in this paper has the added advantage that we can consistently compare between different types as we asymptotically approach the time singularity. In this case the relative strength of the functions describing the singularity type becomes an important factor for distinguishing between the possible behaviours. In this way we can have a clear picture of how "strong" a singularity can be when formed during the evolution and also why such a type eventually arises in any particular model. This charters the singularities in the isotropic category, a necessary first step in an attempt to consider the same classification problem in more complex situations.

It is natural to consider the extension of the work done in this paper in the context of the more general anisotropic Bianchi models. We believe that an analysis of this more complicated case is still feasible using the techniques of the present paper. In such contexts the topology and spatial extent of the singular surfaces are expected to play a role even for the simplest non-trivial 'vacua' such as the Kasner or the Taub–NUT solutions and a complete elucidation of these cases is a prerequisite for situations with matter fields. We leave such matters to future work.

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